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A JOINT CHARACTERIZATION OF GERMAN MONETARY POLICY AND THE DYNAMICS OF THE GERMAN TERM STRUCTURE OF INTEREST RATES

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The paper develops an empirical no-arbitrage Gaussian affine term structure model to explain the dynamics of the German term structure of interest rates. In contrast to most affine term structure models two risk factors are linked to observable macroeconomics factors: output and inflation. The results indicate that the dynamics of the German term structure of interest rates can be sufficiently explained by expected variations in those macroeconomic factors plus an additional unobservable factor. Furthermore, we are able to extract a monetary policy reaction function within this no-arbitrage model that closely resembles empirical reaction functions that are based on the dynamics of the short rate only.

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INTRODUCTION

The characterization of monetary policy by empirical reaction functions that describe the short-term interest rate setting of central banks in response to the dynamics of a small set of macroeconomic variables, e.g. the inflation gap and the output gap, has become very popular in the macroeconomic literature in recent years. Such Taylor-type rules, however, only care about the dynamics of the short-term interest rate. They normally do not take into account the associated dynamics of longer-term interest rates, i.e. the term structure of interest rates. These dynamics are well described by a popular strand of the finance literature, that has also been growing rapidly in recent years. These are no-arbitrage models of the term structure of interest rates. The so-called affine term structure models (ATSM) are the most popular ones within this literature. They explain the dynamics of the term structure of interest rates by (mostly) unobserved stochastic (risk) factors, while yields of different maturities are connected by the absence of arbitrage opportunities.

Although a vast variety of affine term structure models exists due to the number of latent factors and the explicit formulation of their stochastic processes, they all share a common feature: in the single-factor case the only risk factor equals the short rate, whereas in multi-factor cases

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the short rate is a (additive) combination of multiple risk factors. Monetary policy rules share the same (functional) structure, once the risk factors are interpreted as macroeconomic variables. Therefore, the short-term interest rate is a critical point of intersection between the two lines of research. Combining the two lines of research could sharpen our understanding of the dynamics of the term structure of interest rates, respectively, the yield curve.

This paper proposes an affine term structure model that is able to capture the dynamics of the German term structure of interest rates very well and additionally generates reaction coefficients in the short rate equation that are very close to the magnitudes that are observed in empirical models that abstract from arbitrage-free dynamics of the term structure of interest rates. However, in the formulation of the model we face a trade-off: we want to sufficiently characterize the dynamics of the term structure of interest rates but keep the model as tractable as possible, *i.e.*, keeping the number of parameters as low as possible. For this reason we restrict the analysis to the class of constant volatility models with time-invariant term premia.

Several other papers have recently also combined the two lines of research. Ang and Piazessi (2003) estimate the dynamics of the US yield curve also based on macroeconomic factors as well as unobservable factors. They show that macroeconomic factors primarily explain movements at the short end and the middle of the yield curve while unobservable factors account mostly for movements at the long end. However, they do not jointly estimate the coefficients of the macroeconomic factors and the latent factors. They first estimate the coefficients of the monetary policy rule and estimate the remaining parameters in a second step holding the pre-estimated parameters constant. Rudebusch and Wu (2003) estimate a two-factor Gaussian term structure model for the U.S. Based on a subsequent OLS analysis they show that the extracted factors are correlated with factors that typically enter a monetary policy rule.

Our analysis draws on the study of Cassola and Luis (2003) who show that the dynamics of the German term structure from 1972 through 1998 can be well explained by a two-factor Gaussian model with time-invariant risk premia. We add explicit economic content to their model. Hördahl *et al*., (2006) also construct a joint model of German macroeconomic factors and the yield curve. However, they estimate the macroeconomic and the yield curve dynamics within the framework of the new neo-classical synthesis. Our approach lacks such a structural model but has the advantage to link macroeconomic and latent term structure factors more explicitly.

The paper is structured as follows. The subsequent section introduces a discrete affine term structure model based on the general formulation of Duffie and Kan (1996). Section 3 relates the literature of monetary policy rules to this model. The estimation procedure based on a Kalman filter and the results are presented in section 4. Finally, section 5 concludes.

A GAUSSIAN AFFINE TERM STRUCTURE MODEL IN DISCRETE TIME

The term structure of interest rates can be characterized by affine term structure models. These models are based on an explicit no-arbitrage condition in financial markets. Excellent overviews are provided by Maes (2004) and Piazzesi (2007). Tractability is the main advantage of affine models: they assume bond yields to be affine (*i.e.*, constant-plus-linear) functions of some state vector (the risk factors). The one-factor models of Vasicek (1977) and Cox, Ingersoll and Ross

(1985) are the pioneers of that class of models. Duffie and Kan (hereafter DK, 1996) present an unifying framework and establish the conditions that formally preserve the affine property of term structure models. Most existing affine term structure models can thus be interpreted as a particular parameterization of this framework. Among them are multi-factor versions of Vasicek (1977) and Cox, Ingersoll and Ross (1985), as well as Longstaff and Schwartz (1992), Langetieg (1980) and Balduzzi *et al*., (1996 and 1998). Dai and Singleton (2000) provide a categorization of affine term structure models and also formulate the restrictions that have to be fulfilled in the parameterization of the models in order to be theoretically admissible and econometrically identified.

Originally the DK-framework was formulated in continuous time. It has been reformulated by Backus *et al*. (1998a/b) into discrete time. Their model is based on the fundamental result of the existence of an unique pricing kernel. This essentially means that in the absence of arbitrage the price of any financial asset corresponds to the present value of its expected future cash flow with the present value being obtained by applying the positive stochastic discount factor (the pricing kernel). In the case of zero coupon bonds whose future cash flows only correspond to their price in the next period, we can establish the following pricing equation:

$$P_{n,t} = E_t[M_{t+1} P_{n-1,\,t+1}]. \tag{2.1}$$

Here, P represents the price of the zero coupon bond with n denoting its time to maturity. M is the stochastic discount factor which is also known as the price generator, since prices grow from it. In any arbitrage-free environment, there exists an unique positive random variable that satisfies the above expression (see Harrison and Kreps (1979) as well as Harrison and Pliska (1981)). In consumption-based equilibrium models, the pricing kernel represents the marginal rate of substitution between present and next period's consumption, *i.e*., the discounted ratio of marginal utilities of consumption. Arbitrage opportunities are ruled out by applying the same discount factor to all bonds. Solving forward the basic pricing equation (2.1) by the *law of iterated expectations* and noting that the bond pays exactly one unit at maturity ($P_{0,t+n} = 1$) yields:

$$P_{n,t} = E_t[M_{t+1} \dots M_{t+n}] = E_t\left[\prod_{i=1}^{n} M_{t+i}\right], \tag{2.2}$$

so that a model of bond prices could also be expressed as a model of the evolution of the pricing kernel. It follows that we can model $P_{n,t}$ (and, thus, associated bond yields) by modeling the stochastic process of M_{t+i}. The bond prices (and, thus, bond yields) are a function of those state variables that are relevant for forecasting the process of the pricing kernel, *i.e*., the process of the underlying risk factors. In particular, the k-dimensional vector of risk factors, Z, satisfies the following stochastic process:

$$Z_{t+1} = (I - \Phi) \cdot \theta + \Phi \cdot Z_t + V(Z_t)^{1/2}\, \varepsilon_{t+1} \tag{2.3}$$

where I is an identity matrix and the matrix Φ has positive diagonal elements between zero and one in order to ensure that the factors are stationary. θ is the long-run mean of the risk factors. Thus, the risk factors are governed by a discrete-time *Ornstein-Uhlenbeck* process. The independent shock term is normally distributed with $\varepsilon_t \sim N(0, I)$. Finally, $V(Z_t)$ is the variance-covariance matrix of the random shocks and is defined as a diagonal matrix with the elements $v_j(Z_t) = \alpha_j +$

$\beta_j' Z_t$, where β_j has nonnegative elements. Thus, the factors have an affine volatility structure, which is a generalization of the square-root structure of the Cox, Ingersoll and Ross (1985) model. Moreover, the factors are allowed to be correlated. The square-root process of the risk factor requires that the volatility function $v_j(Z)$ has to be positive, which places particular restrictions on the parameters. According to the indexation, please note that in the subsequent notation the subscripts j generally indicate a particular element within the respective matrix which has no subscript attached. Throughout the paper the subscript $j = 1 \ldots k$ indicates the number of risk factors, and thus, determines the size of the matrices and vectors.

To derive an affine yield model, the distribution of the stochastic discount factor is assumed to be conditionally log-normal. In addition to providing model tractability, this assumption keeps the discount factor positive and unique. Following Backus *et al*. (1998a), the negative of the log of the pricing kernel ($m_{t+1} \equiv \log[M_{t+1}]$) takes the form:

$$-m_{t+1} = \delta + \gamma' \cdot Z_t + \lambda' \, V(Z_t)^{1/2} \, \varepsilon_{t+1}, \tag{2.4}$$

where δ is a constant, γ is a parameter vector and λ is the vector that governs the correlation between innovations in the risk factors (state variables) and the pricing kernel: risk, in other words.

According to the affine formulation of the model it is assumed that bond prices are exponential affine functions of the risk factors. This particular formulation ensures the property that log bond prices, and hence bond yields, are linear (affine) in the state variables. This ensures the desired joint log-normality of bond prices with the stochastic discount factor:

$$P_{n,t} = \exp(-A_n - B_n' \cdot Z_t), \tag{2.5}$$

where, again, the second subscript n denotes the time to maturity at time t. The parameter A_n and the vector of parameters B_n are to be estimated in order to determine bond prices. Both are 'time to maturity-related' constants. The latter is commonly referred to as the vector of factor loadings because it measures the impact of a shock to the risk factors on the bond prices. In case of zero coupon bonds the values A_0 and B_0 have to be equal to zero, because at the time of their maturity ($n = 0$) they pay exactly the face value of one unit by definition. Thus, the log of the price of a maturing bond (denoted by a lower case p) has to be zero. The general form of the relation is:

$$-p_{n,t} = A_n + B_n' \cdot Z_t. \tag{2.6}$$

Since nominal yields and prices of zero coupon bonds are unambiguously linked, nominal yields of zero coupon bonds can be easily computed as

$$i_{n,t} = -\frac{p_{n,t}}{n} = \frac{A_n}{n} + \frac{B_n'}{n} Z_t \,. \tag{2.7}$$

Equation (2.7) states that we can model the dynamics of the whole term structure if we are able to estimate the parameters A_n and the vector B_n for all values of n.

In order to work with a tractable model we propose a particular parameterization of the general model sketched out above. The parameter choice is driven by the following trade-off. On the one hand, the model has to be sufficiently flexible to be able to characterize the dynamics of the German term structure of interest rates. This might call for a fully fledged model with a high number of parameters to be estimated. On the other hand, our interest lies in keeping the number of parameters as low as possible, in order to keep the model empirically tractable and also to assign specific economic interpretation to the estimated parameters. In order to resolve this trade-off, we draw on the study of Cassola and Luis (2003). These authors recently demonstrated that the dynamics of the German term structure of interest rates can be well explained within a constant volatility (or Gaussian) two-factor affine term structure model. Our choice of parameters is close to their parameterization. It is as follows:

$$\begin{aligned} \theta_j &= 0 \\ \Phi &= \operatorname{diag}(\varphi_1 \ldots \varphi_k) \\ \alpha_j &= \sigma_j^2 \\ \beta_j &= 0 \\ \delta &= \bar{\delta} + \frac{1}{2}\lambda'\sigma\sigma'\lambda\,. \end{aligned} \tag{2.8}$$

The long-run mean of the risk factors is set equal to zero because of our particular identification assumption of the risk factors that follows below. Additionally, there is general technical justification for this as well. Backus *et al*. (1998a) show for the single-factor case of the constant volatility model the parameters δ and θ cannot be identified separately, so that one of them could effectively be dropped. Setting either δ or θ to zero implies identical asset prices. However, in the general setup of the model this is not the case. The matrix of the mean reversion parameters is assumed to be diagonal so that the dynamics of a particular risk factor only depend on its own current value relative to its long-run mean. The third and fourth definition in (2.8) constitute a Gaussian model with constant volatility of the risk factors so that the variance-covariance matrix reduces to the diagonal matrix σ with the elements σ_j. The last definition in (2.8) is intended to yield a particular formulation for the short-rate equation. In particular, this variance term only arises because we are working in logs (*Jensen's Inequality*). The normalization lets us get rid of this term. In contrast to Cassola and Luis (2003) we limit the number of restrictions put on the parameters and admit the parameter vector γ to be determined by the data.

The parameterization yields the following equation for the dynamics of the pricing kernel:

$$-m_{t+1} = \bar{\delta} + \frac{1}{2}\lambda'\sigma\sigma'\lambda + \gamma' \cdot Z_t + \lambda'\sigma\varepsilon_{t+1} \cdot \tag{2.9}$$

The vector of risk factors follows a first-order autoregressive process:

$$Z_{t+1} = \Phi \cdot Z_t + \sigma\varepsilon_{t+1}\,, \tag{2.10}$$

Combining the fundamental pricing relation in (2.1) expressed in logs together with eqs. (2.6), (2.9), (2.10), the known (zero) values A_0 and B_0, and the assumption of the independence of shocks the values of A_n and B_n will follow the subsequent recursive restrictions (see Appendix I for the derivation):

$$A_{n+1} = A_n + \bar{\delta} + \frac{1}{2}\lambda'\sigma\sigma'\lambda - \frac{1}{2}(\lambda' + B_n')'\,\sigma\sigma'\,(\lambda' + B_n') \tag{2.11}$$

$$B_{n+1}' = (\gamma' + B_n'\Phi). \tag{2.12}$$

From the recursive restrictions (2.11) and (2.12) together with (2.7), and the fact that $A_0 = B_0 = 0$, we can get an expression for the one-period interest rate:

$$i_{1,t} = \overline{\delta} + \sum_{j=1}^{k} \gamma_j z_{j,t}. \tag{2.13}$$

The one-period (or short-term) interest rate is determined by a constant and the sum of time-varying risk factors multiplied by related (constant) coefficients. Examining (2.13) and interpreting the one-period interest rate as the policy rate of a central bank, one immediately realizes its equivalence to the popular class of empirical monetary policy rules, in which the risk factors and their associated coefficients would have very specific economic meanings. In particular, the constant would be the equilibrium level of the nominal short-term interest rate, the γ_j's would resemble the reaction coefficients of the central bank and the risk factors would display macroeconomic variables upon which the central bank reacts. From an empirical point of view equation (2.13) potentially allows us to jointly estimate a monetary policy reaction function together with the arbitrage-free dynamics of the term structure of interest rates. However, the crucial point for this exercise is the (empirical) identification of the risk factors. Cassola and Luis (2003), for example, parameterize the model such that the one-period rate is just the sum of risk factors. In the two-factor case they notice that this equals the *Fisher equation*, where the nominal interest rate is the sum of the real interest rate plus expected inflation. We will take another avenue as described in the subsequent section.

THE MONETARY POLICY VIEW OF THE SHORT RATE DYNAMICS

Since the influential paper by Taylor (1993), it has become common practice to model monetary policy as a simple feedback or instrument rule. These rules link the short term policy rate to measures of the output gap and the inflation gap:

$$i_{1,t} = \pi_t + \overline{r} + \gamma_\pi(\pi_t - \pi^*) + \gamma_y(y_t - \overline{y}_t), \tag{3.1}$$

where i_1 is the policy rate, π is inflation with the (*) indicating its target value. The equilibrium real interest rate is $\overline{r}$ and y is an output measure with the upper bar indicating its equilibrium level. The coefficients γ_π and γ_y measure the strength of the reaction to the inflation gap, respectively, the output gap. A comparison of (3.1) and (2.13) reveals the same expression, once the sum of the equilibrium real rate and the actual inflation rate reflect the $\overline{\delta}$ and the (two) risk factors are interpreted as the inflation and the output gap.

Clarida *et al*., (1998) generalize the formulation in (3.1) to a class of explicitly forward-looking instrument rules. They further allow for the effect that central banks smooth their changes in the policy rate, so that the actual rate adjusts only partially to its target rate. An alternative interpretations for presence of a lagged interest rate value is the serial correlation of shocks (Rudebusch (2002). The empirical representation of the reaction function then becomes:

$$\begin{aligned} i_{1,t} = {} & (1-\rho)(\overline{r} + E_t(\pi^*_{t+q} | I_t) + \gamma_\pi E_t(\pi_{t+q} - \pi^*_{t+q} | I_t) \\ & + \gamma_y E_t(y_{t+l} - \overline{y}_{t+l} | I_t)) + \rho \cdot i_{1,t-1} + v_t, \end{aligned} \tag{3.2}$$

where ρ represents the smoothing parameter, E_t is the (conditional) expectation operator given the information set I_t, q and l are the horizons of the forward-looking behavior, and ν is a composite error term that comprises exogenous shocks to the policy rate. Comparing (3.2) and (2.13) would then relate two risk factors with the expected inflation gap, the expected output gap as well as a third risk factor that comprises the effect of interest rate smoothing behavior and other (stochastic) shocks to the interest rate.

In order to explicitly combine the forward-looking monetary policy rule representation with the arbitrage-free model of the term structure, we need to characterize the dynamics of the inflation and the output process. In particular, we assume that the inflation gap and the output gap both follow an AR(1) process:

$$\tilde{\pi}_{t+1} = \chi_\pi \cdot \tilde{\pi}_t + u_{t+1}, \tag{3.3}$$

$$\tilde{y}_{t+1} = \chi_y \cdot \tilde{y}_t + \upsilon_{t+1}, \tag{3.4}$$

where the $\tilde{x}$-notation reflects the gap of the respective variable. Forward iteration of (3.3) yields:

$$\tilde{\pi}_{t+q} = \chi_\pi^q \cdot \tilde{\pi}_t + \sum_{i=1}^{q} \chi^{q-i} u_{t+i}. \tag{3.5}$$

If we match one of the risk factors in time t with the expected inflation gap $t + q$ periods ahead

$$z_{\pi,t} = E_t[\tilde{\pi}_{t+q}] = \chi_\pi^q \cdot \tilde{\pi}_t, \tag{3.6}$$

the law of motion of this (inflation) risk factor becomes

$$z_{\pi,t+1} = E_t[\tilde{\pi}_{t+q+1}] = \chi_\pi \cdot \tilde{\pi}_{t+q} = \chi_\pi(\chi_\pi^q \cdot \tilde{\pi}_t) + \chi_\pi\left(\sum_{i=1}^{q} \chi^{q-i} u_{t+i}\right). \tag{3.7}$$

Combining (3.6) and (3.7) with the law of motion for the risk factors from the affine term structure model in (2.10) it follows for the identification of one parameters that:

$$\varphi_\pi = \chi_\pi$$

$$\sigma_\pi \varepsilon_{t+1} = \chi_\pi \cdot \sum_{i=1}^{q} \chi_\pi^{q-i} u_{t+i}. \tag{3.8}$$

From (3.6) it follows that the actual inflation gap and the actual (inflation) risk factor are connected as follows:

$$\tilde{\pi}_t = \frac{1}{(\varphi_\pi)^q} \cdot z_{\pi,t}. \tag{3.9}$$

By the same token the relationship between the actual output gap and the (output) risk factor is:

$$\tilde{y}_t = \frac{1}{(\varphi_y)^l} \cdot z_{y,t}. \tag{3.10}$$

Equations (3.9) and (3.10) follow from the generalized monetary policy rule in (3.2) and display the general identification of two risk factors with observable macroeconomic variables given the assumption of an AR(1) behavior of the latter. The Taylor rule in (3.1) is the special case with $q = l = 0$. Equations (3.9) and (3.10) when combined with (2.7) provide us with the possibility to jointly estimate the yields and the two macroeconomic factors as functions of the risk factors.

ESTIMATION PROCEDURE AND RESULTS

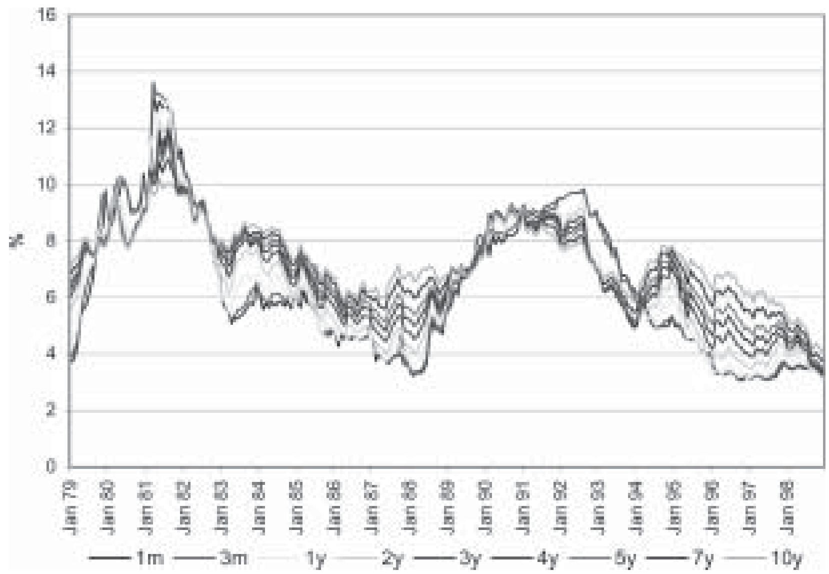
We focus on German data. The Appendix II provides details of data sources and handling. The data are monthly and cover the period from 1:1979 to 12:1998. This period represents what we might call the 'Bundesbank regime'. The sample starts in 1979 because this represents the starting of the European Exchange Rate Mechanism and ends in 1998, when the European Central Bank took over responsibility for monetary policy in the euro area. The estimates of the German yield curve are based on the parameters of the Svensson (1994) smoothing technique as made available by the Bundesbank. The data set comprises monthly averages of nine daily spot rates for the following maturities (in month): 1, 3, 12, 24, 36, 48, 60, 84 and 120. Because of the somewhat erratic behavior we substitute the 'Svensson estimates' of the one month and three months interest rates by the observed German interbank rates. Figure 1 presents the time series properties of the yields while Table 1 presents the summary statistics.

Table 1
Statistical Properties of German Government Bond Yields 1979-1998

Maturity	*1*	*3*	*12*	*24*	*36*	*48*	*60*	*84*	*120*
Mean	6.321	6.343	6.376	6.569	6.762	6.921	7.048	7.226	7.376
St.Dev	2.499	2.539	2.30	2.078	1.899	1.746	1.616	1.416	1.215
Skewness	0.604	0.599	0.544	0.379	0.316	0.287	0.259	0.167	0.031
Kurtosis	2.487	2.537	2.636	2.408	2.398	2.472	2.571	2.755	2.955
Autocorr.	0.982	0.984	0.982	0.980	0.978	0.976	0.973	0.968	0.960
Correlation Matrix									
1	1.0000	0.9967	0.9724	0.9399	0.9132	0.8914	0.8715	0.8319	0.7678
3		1.0000	0.9826	0.9543	0.9294	0.9084	0.8890	0.8500	0.7870
12			1.0000	0.9902	0.9755	0.9606	0.9453	0.9119	0.8552
24				1.0000	0.9959	0.9872	0.9760	0.9489	0.9009
36					1.0000	0.9974	0.9909	0.9708	0.9307
48						1.0000	0.9980	0.9850	0.9525
60							1.0000	0.9938	0.9689
84								1.0000	0.9899
120									1.0000

Table 1 reveals that yields are highly persistent with monthly autocorrelations above 0.96 for all maturities. The volatility drops with increasing maturity. Also yields are highly correlated along the yield curve but the correlations are not equal to one. This suggests that non-parallel shifts of the yield curve are an important feature. Indeed, the yield curve frequently changed its slope, shape and curvature and even periods of inverse yield curves can be identified. Therefore, a one-factor model is insufficient for explaining the dynamics of the German term structure of interest rates.

Figure 1: Term Structure Dynamics 1979-1998



The data on the inflation gap is constructed using the difference of the consumer price index and the so-called price norm that has been announced by the Bundesbank on a yearly basis. This could be interpreted as a kind of an inflation target, although the Bundesbank did not explicitly target inflation but the money growth rate. Nevertheless, the inflation values were used as inputs into the derivation of the money growth target and price stability was the final target of the Bundesbank. See Appendix II for more information on the price norm.

The output gap is based on the monthly index of industrial production. Although being aware that industrial output is more volatile than, say, GDP, we follow the usual practice and opt for this variable in order to obtain monthly figures. The output gap is constructed on the basis of a linear trend. In order to avoid starting point and end point problems we computed the trend between the longer period from 1970 through 2003. We also used a HP filter and the reported results were robust against this change. Furthermore, it turned out that a quadratic trend did not change our measure of the output gap, since the quadratic term turned out to be insignificant. Thus, we opted for the linear trend.

We first present the results of standard regressions of monetary policy reaction function on the basis of our data sample, where the short-term interest rate is the German overnight money market rate. Table 2 shows the results of two regressions. The first is based on equation (3.1). This standard Taylor rule provides a quite good fit for our sample period. The coefficients of the output gap and the inflation gap are significant, and the latter fulfills what is known as the Taylor-principle. The coefficient of the output gap is rather low but significant.

In order to estimate the forward-looking monetary policy rule in equation (3.2), we follow the usual procedure and implement a GMM estimation. We depart from Clarida, Galí and Gertler (1998) and set $q = 12$ and $l = 3$, which yields a better fit compared to the specification without a forward-looking output gap. In the regression, we use the following set of instruments with their lag structure being described in the Appendix II: a constant as well as lagged values of the output gap, the inflation gap, the IMF commodity price index and the short-term interest rate. The results are presented in Table 2. In the forward-looking specification, the coefficient of the inflation gap is again significant and of a higher value compared to the plain Taylor rule specification. The coefficient of the output gap is of comparable size. The smoothing parameter is quite high but is in the expected range. In the next section we will use these results as a benchmark for the joint estimation of the monetary policy rule coefficients and the term structure dynamics. It should already be noted that in this term structure estimation we marginally deviate from the specification above. Since the one month rate is the shortest maturity in the sample of yields, we will interpret this as the policy rate. However, the correlation between the overnight money market rate and one month rate in our data set is 0.9951. The results in Table 2 are robust against a change in the dependent variable.

Table 2
Monetary Policy Rule Estimates

	Taylor Rule	*CGG (q = 12, l = 3)*
Constant	5.85	4.06
	(0.105)	(0.34)
γ_π	1.19	1.68
	(0.071)	(0.132)
γ_y	0.17	0.089
	(0.024)	(0.041)
ρ	–	0.89
		(0.017)
R^2	0.62	0.98

Standard errors in parentheses

We next present the joint estimation of monetary policy rule coefficients and the term structure dynamics based on our affine term structure model. The estimation relies on a Kalman-filter-based maximum likelihood estimator. The Kalman filter approach has gained popularity in the affine term structure literature. See, for example, Lund (1997), Gong and Remolona (1997), Babbs and Nowman (1999), De Jong (2000), Bolder (2001) and Cassola and Luis (2003). This approach allows us to estimate the parameters of the model without directly observing the risk factors. However, we suspect that two of the factors are connected to an inflation and an output measure through eqs. (3.9) and (3.10). In order to apply a Kalman filter we first have to write the model in the linear state-space form, with the measurement equation resulting from the recursive restrictions and the identification of the factors, and the transition equation resulting from the assumed dynamics of the risk factors.

In particular, in the '3-factor partly-identified' case, where two factors are linked to the inflation gap and the output gap, and the third factor is unobservable, the measurement equation is:

$$\begin{bmatrix} i_{1,t} \\ i_{3,t} \\ \vdots \\ i_{120,t} \\ \tilde{\pi}_t \\ \tilde{y}_t \end{bmatrix} = \begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{120} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \\ \vdots & \vdots & \vdots \\ b_{120,1} & b_{120,2} & b_{120,3} \\ 1/(\varphi_\pi)^q & 0 & 0 \\ 0 & 1/(\varphi_y)^l & 0 \end{bmatrix} \cdot \begin{bmatrix} z_{\pi,t} \\ z_{y,t} \\ z_{3,t} \end{bmatrix} + \begin{bmatrix} \nu_{1,t} \\ \nu_{3,t} \\ \vdots \\ \nu_{120,t} \\ \nu_{\pi,t} \\ \nu_{y,t} \end{bmatrix}, \qquad (4.1)$$

with $a_i = A_i / i$ and $b_{i,j} = B_{i,j} / i$, where the index i represents the number of months to maturity (i = 1, 3, 12, 24, 36, 48, 60, 84, 120) and j the index of the (up to three) risk factors. Expression (4.1) shows that we estimate a system of equations, *i.e.*, a panel for different yields to maturity, with the additional interpretation that the first equation is the monetary policy reaction function of the central bank. Or, from a different angle, we can interpret our exercise as estimating a monetary policy rule (corresponding to the first rows in the matrixes in (4.1)) with the additional restriction of the absence of arbitrage in financial markets. The corresponding transition equation is represented by:

$$\begin{bmatrix} z_{\pi,t+1} \\ z_{y,t+1} \\ z_{3,t+1} \end{bmatrix} = \begin{bmatrix} \varphi_\pi & 0 & 0 \\ 0 & \varphi_y & 0 \\ 0 & 0 & \varphi_3 \end{bmatrix} \cdot \begin{bmatrix} z_{\pi,t} \\ z_{y,t} \\ z_{3,t} \end{bmatrix} + \begin{bmatrix} \sigma_\pi & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{\pi,t+1} \\ \varepsilon_{y,t+1} \\ \varepsilon_{3,t+1} \end{bmatrix} \tag{4.2}$$

In the ‘2-factor identified’ case, the state-space system is written without the third factor leaving the identification for the two remaining factors unchanged. Additionally, we estimate a ‘2-factor non-identified’ case that resembles a traditional affine term structure model with only unobserved factors.

Table 3 presents the results of the three specifications. The first specification is the ‘2-factor non-identified’ case. The second is the 2-factor model with both factors being identified with

Table 3
Parameter Estimates of the Gaussian Term Structure Model

	2F_non_ident	*2F_ident*	*3F_ident*
$\varphi_1 (\varphi_y)$	0.87	0.82	0.98
	(0.0025)	(0.0047)	(0.001)
$\varphi_2 (\varphi_\pi)$	0.99	0.98	0.99
	(0.0004)	(0.0003)	(0.002)
φ_3	–	–	0.89
			(0.0058)
$\sigma_1 (\sigma_y)$	0.006	0.0158	0.00406
	(0.0001)	(0.0002)	(0.0001)
$\sigma_2 (\sigma_\pi)$	0.0016	0.0007	0.00045
	(0.0001)	(0.00001)	(0.0001)
σ_3	–	–	0.00286
			(0.0001)
$\lambda_1 (\lambda_y)$	–1.364	7.19	1.743
	(0.2456)	(0.2317)	(1.55)
$\lambda_2 (\lambda_\pi)$	–52.16	–113.89	–139.39
	(0.6664)	(0.8467)	(4.69)
λ_3	–	–	–3.688
			(4.81)
δ	0.0038	0.0043	0.0043
	(0.0001)	(0.0001)	(0.0001)
$\gamma_1 (\gamma_y)$	1.363	0.092	0.088
	(0.0287)	(0.0027)	(0.0153)
$\gamma_2 (\gamma_\pi)$	0.934	2.374	1.764
	(0.0240)	(0.0369)	(0.0469)
γ_3	–	–	2.566
			(0.116)

Standard errors in parentheses

the expected inflation and output gap respectively ('2-factor identified'). Since the '2-factor identified' specification might be very restrictive, we also estimated the '3-factor partly-identified' specification in which two factors are identified as in the previous case and the third factor being unobserved. The last one corresponds to the state space system in (4.1) and (4.2). In both identified specifications we set $q = 12$ and $l = 3$ as we did in the benchmark estimation of the forward-looking specification of the monetary policy rule.

Especially in the '3-factor partly-identified' specification, the reaction coefficients are close to the levels that we obtained in the estimation of the forward-looking monetary policy rule before. This indicates that the estimated policy rule is consistent with the absence of arbitrage opportunities in financial markets. The reaction coefficient, the volatility and the coefficient of mean reversion of the third unobserved factor are also significant but its associated market price of risk is not. If the third factor includes the smoothing behavior, this result seems quite natural because there is little risk associated with this factor. Thus, the result suggests that the risk premia in the German term structure of interest rates mainly stems from the inflation and the output risk. Note that only the inflation risk price is negative while the output risk price is positive.

The mean-reversion coefficients of inflation and output are quite high but close to the coefficients we obtained in a pure $AR(1)$ regression of both. These values for the sample are 0.97 (0.91) for the inflation gap (output gap) with a R^2 of 0.93 (0.83). This suggests that our identification based on a simple $AR(1)$ process is not too far from reality and seems consistent with the data.

Figure 2 shows the estimated dynamics of the term structure for the preferred '3-factor partly-identified' case. It reveals that the model captures the overall movement in the term structure that is shown in Figure 1 very well.

Figure 2: Estimated Term Structure Dynamics

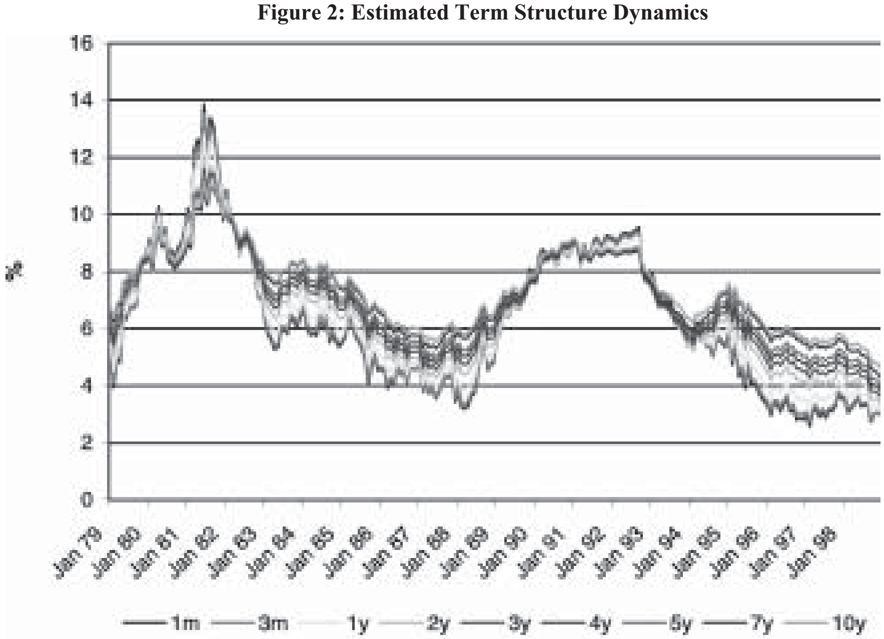


Figure 3 shows the estimated and observed yields for three particular maturities that represent the short end, the middle range and the long end of the yield curve. It shows that the model especially has a good fit at the middle range and the short end. This might not surprise since we model time-constant (*i.e.*, only maturity dependent) risk premia.

Figure 3: Observed and Estimated Yields for Particular Maturities

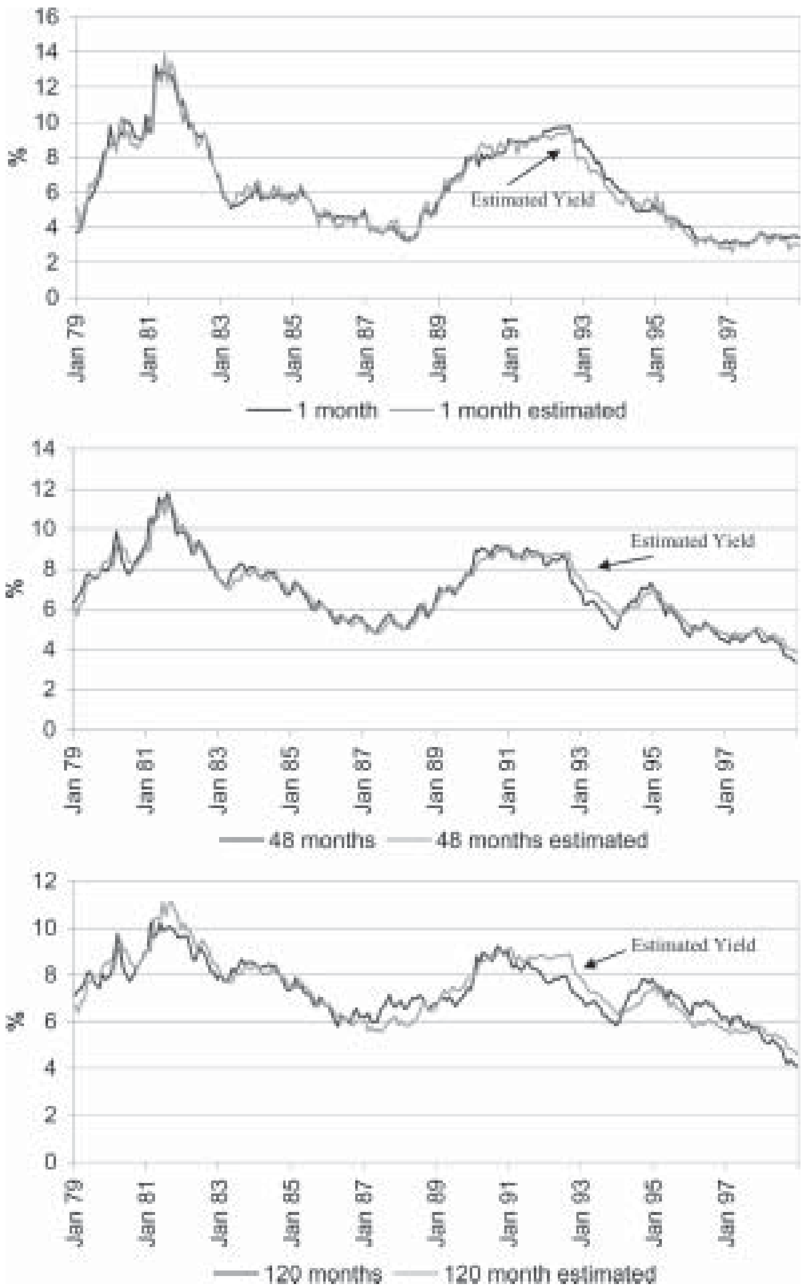
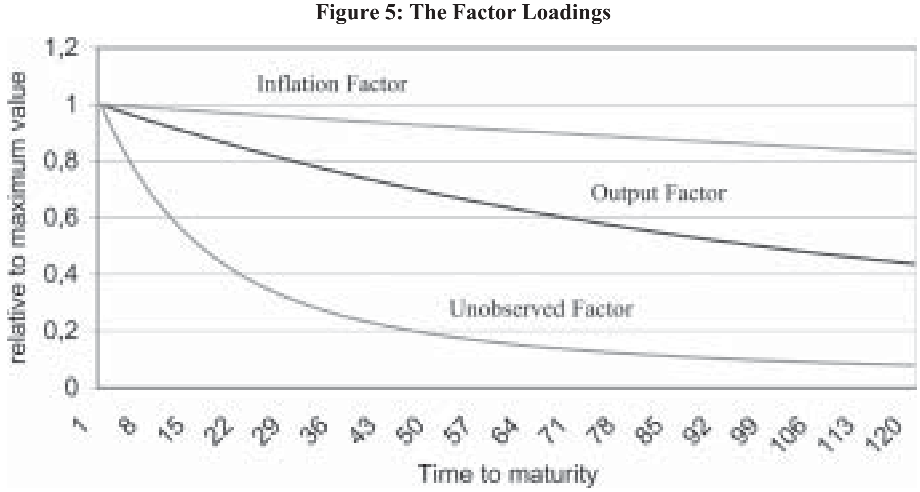


Figure 4 answers the question whether the model is able to capture the periods of an inverse term structure of interest rates. It shows the observed and the estimated spread defined as the yield of 10-year bonds minus the yield of one-month bonds. Again the overall fit is good.

Figure 4: Observed and Estimated Spread (10y – 1m)

Figure 5 presents the loadings that are associated with each factor in the '3-factor partly-identified' case. It shows that the 'inflation factor' has a nearly equal impact on all maturities. In a standard interpretation we would consider this as the 'level factor'. This interpretation seems quite reasonable given the high persistence of this factor as well as the fact that it incorporates the inflation target of the central bank, and thus the long-run expectations of inflation.

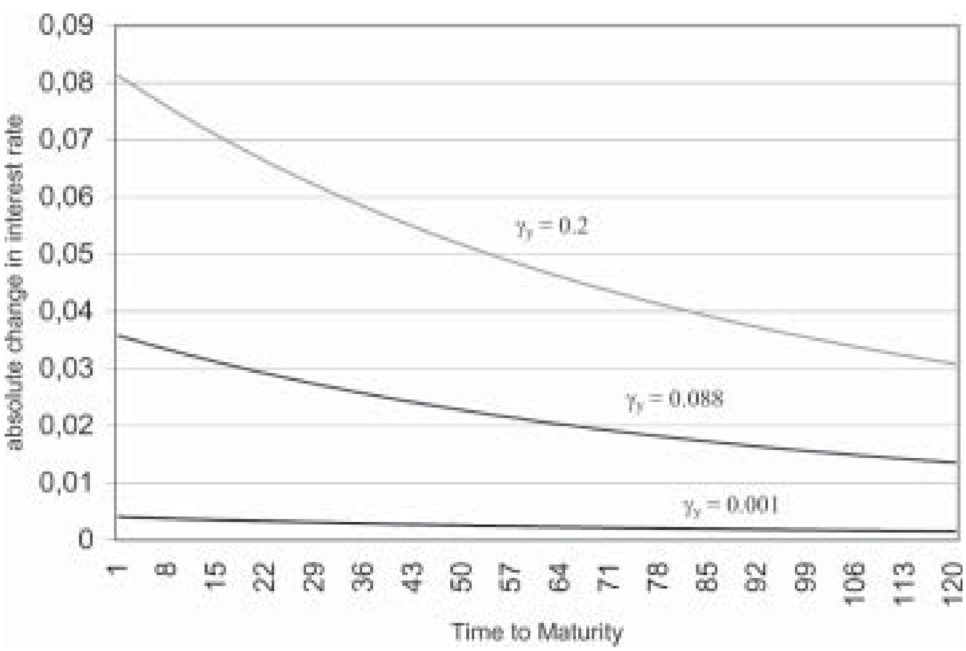
Figure 5: The Factor Loadings



Through the *Fisher relation*, this results in a long-run level of interest rates. The impact of the 'output factor' declines stronger along the yield curve. This is consistent with the interpretation as the 'curvature factor'. The third unobserved factor has its major impact on the short end of the yield curve. This is in line with the third 'smoothing factor' in monetary policy rules that incorporate interest rate smoothing behavior which has its mean impact on the short end of the yield curve. Independent of its precise interpretation, the third factor can be considered as the 'slope factor'. We could also interpret this factor as capturing the forecast error that emerges in the error term of the forward looking specification of the rule.

From (2.12) we see that the monetary policy reaction coefficients also influence the factor loadings. However, they do not influence the relative impact of a shock to the particular risk factor along the yield curve as indicated in Figure 5. The strength of the policy reaction to a given shock rather determines the absolute impact of a shock on the yield curve. For instance, a stronger output response *ceteris paribus* leads to greater importance of the 'curvature factor' relative to the other factors, meaning that it is more likely that the yield curve changes its curvature. In this sense, monetary policy reaction determines to what extent given shocks alter the yield curve. Figure 6 shows the change in interest rates along the yield curve due to a shock to the output factor of the size of one standard deviation. The stronger the policy reaction to this given shock is, the greater is the absolute difference between the short rate and the long rate change, thus, the higher the effect on the curvature of the yield curve.

Figure 6: The Influence of the Policy Reaction



CONCLUSIONS

The paper constructed a Gaussian affine term structure model with a clear economic underpinning of the factors that drive the dynamics of the German term structure of interest rates. It shows that matching two out of three factors with the expected inflation gap and the expected output gap yields a reasonable empirical characterization of the German yield curve between 1979 and 1998. Furthermore, the empirical results are consistent with the results obtained from a forward-looking monetary policy reaction function that only cares about the dynamics of the short-term interest rate, or the policy rate. However, the monetary policy reaction function that is extracted from the affine term structure model has the additional feature that it is consistent with the absence of arbitrage in financial markets.

Still, there are some avenues for future research to improve our analysis. First, the possibility of time-varying term premia should improve our results. This is commonly modeled by defining the risk premia as a linear function of the risk factors. Second, an improvement of the identification of the risk factors is preferable, since the $AR(1)$ process for the inflation and the output gap, though being a good first approximation, seems too restrictive. Third, other macroeconomic factors such as foreign interest rates, monetary aggregates or the exchange rate might as well influence the dynamics of the term structure of interest rates. Future research should aim to account for these additional influences.

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REFERENCES

Ang A., and M. Piazessi, (2003), "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables", *Journal of Monetary Economics*, **50**, 745-787.

Babbs S. H., and K. B. Nowman, (1999), "Kalman Filtering of Generalized Vasicek Term Structure Models", *Journal of Financial and Quantitative Analysis*, **34**, 115-130.

Backus D., Foresi S., and C. Telmer, (1998a), "Discrete Models of Bond Pricing", NBER Working Paper No. 6736, Cambridge/MA.

Backus D., Foresi S., and C. Telmer, (1998b), "Affine Models of Currency Pricing: Accounting for the Forward Premium Anomaly", Mimeo, Stern School of Business.

Balduzzi P., Das S., and S. Foresi, (1998), "The Central Tendency: A Second Factor in Bond Yields", *Review of Economics and Statistics*, **80**, 60-72.

Balduzzi P., Das S., and S. Foresi, (1996), "A Simple Approach to Three-Factor Affine Term Structure Models", *Journal of Fixed Income*, **6**, 43-53.

Bolder D. J., (2001), "Affine Term-Structure Models: Theory and Implementation", Bank of Canada Working Paper 2001-15, Ottawa.

Cassola N., and J. B. Luis, (2003), "A Two-factor Model of the German Term Structure of Interest Rates", *Applied Financial Economics*, **13**, 783-806.

Clarida R., Galí J., and M. Gertler, (1998), "Monetary Policy Rules in Practice: Some International Evidence", *European Economic Review*, **42**, 1033-1067.

Cox J., Ingersoll J., and S. Ross, (1985), "A Theory of the Term Structure of Interest Rates", *Econometrica*, **53**, 385-407.

Dai Q., and K. Singleton, (2000), "Specification Analysis of Affine Term Structure Models", *Journal of Finance*, **55**, 1943-1978.

De Jong F., (2000), "Time Series and Cross-section Information in Affine Term-Structure Models", *Journal of Business & Economic Statistics*, **18**, 300-314.

Duffie D., and R. Kan, (1996), "A Yield Factor Model of Interest Rates", *Mathematical Finance*, **6**, 379-406.

Gong F. F., and E. M. Romolona, (1997), "Two Factors Along the Yield Curve", *The Manchester School*, **65**, (Supplement), 1-31.

Harrison M., and D. Kreps, (1979), "Martingales and Arbitrage in Multiperiod Security Markets", *Journal of Economic Theory*, **20**, 381-408.

Harrison M., and S. Pliska, (1981), "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and Their Application*, **11**, 215-260.

Hördahl P., Tristani O., and D. Vestin, (2006), "A Joint Econometric Model of Macroeconomic and Term Structure Dynamics", *Journal of Econometrics*, **131**, 405-444.

Langetieg T., (1980), "A Multivariate Model of the Term Structure", *The Journal of Finance,* **35**, 71-97.

Longstaff F., and E. Schwartz, (1992), "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model", *The Journal of Finance*, **47**, 1259-1282.

Lund J., (1997), "Econometric Analysis of Continuous-Time Arbitrage-Free Models of the Term Structure of Interest Rates", Manuscript, Department of Finance, The Aarhus School of Business.

Maes K., (2004), "Modeling the Term Structure of Interest Rates: Where Do We Stand?", National Bank of Belgium Working Paper No. 42, Brussels.

Piazzesi M., (2007), "Affine Term Structure Models", In Y. Ait-Sahalia, and L. Hansen, (Eds.) 2007, *Handbook of Financial Econometrics*, Elsevier, North-Holland.

Rudebusch G. D., (2002), "Term Structure Evidence on Interest Rate Smoothing and Monetary Policy Inertia", *Journal of Monetary Economics*, **49**, 1161-1187.

Rudebusch G. D., and T. Wu, (2003), "A Macro-Finance Model of the Term Structure, Monetary Policy, and the Economy", Federal Reserve Bank of San Francisco Working Paper No. 2003-17.

Svensson L. E. O., (1994), "Estimating and Interpreting Forward Interest Rates: Sweden 1992-94", IMF Working Paper 94/114, International Monetary Fund, Washington/DC.

Taylor J. B., (1993), "Discretion Versus Monetary Policy Rules in Practice", *Carnegie Rochester Conference Series on Public Policy*, **39**, 195-214.

Vasicek O., (1977), "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, **5**, 177-188.

APPENDIX I: THE RECURSIVE RESTRICTIONS

The derivation starts with writing the expression (2.1) in logs (for convenience we let the bond in time t mature in $n + 1$ periods):

$$p_{n+1,\,t} = \log\left(E\left\lfloor M_{t+1}\, P_{n,\,t+1}\right\rfloor\right). \tag{A.1}$$

With the assumption of joint log-normality of bond prices and the nominal pricing kernel and using the statistical principle that if $\log X \sim N(\mu, \sigma^2)$ then $\log E(X) - \mu + \frac{\sigma^2}{2}$, it follows that:

$$p_{n+1,\,t} = E_t\,[m_{t+1} + p_{n,\,t+1}] + \frac{1}{2}\, Var_t\,[m_{t+1} + p_{n+1,\,t}]. \tag{A.2}$$

From (2.6) and (2.9) we get

$$m_{t+1} + p_{n,\,t+1} = -\,\bar{\delta} - \frac{1}{2}\,\lambda'\sigma\sigma'\lambda - \gamma' \cdot Z_t - \lambda'\sigma\varepsilon_{t+1} - A_n - B'_n \cdot Z_{t+1}. \tag{A.3}$$

After substituting (2.10) for Z_{t+1} we have

$$E[m_{t+1} + p_{n,\,t+1}] = -\left(A_n + \bar{\delta} + \frac{1}{2}\,\lambda'\sigma\sigma'\lambda + (\gamma' + B'_n\,\Phi) \cdot Z_t\right), \tag{A.4}$$

$$Var[m_{t+1} + p_{n,\,t+1}] = (\lambda' + B'_n)'\sigma\sigma'\,(\lambda' + B'_n). \tag{A.5}$$

Substituting this into (A.2) we get:

$$-\,p_{n+1,\,t} = A_n\;\bar{\delta} + \frac{1}{2}\,\lambda'\sigma\sigma'\lambda + (\gamma' + B'_n\,\Phi) \cdot Z_t - \frac{1}{2}\,(\lambda' + B'_n)'\sigma\sigma'\,(\lambda' + B'_n). \tag{A.6}$$

Matching coefficients in order to verify that

$$-\,p_{n+1,\,t} = A_{n+1} + B'_{n+1} \cdot Z_t \tag{A.7}$$

yields the recursive expressions:

$$A_{n+1} = A_n + \bar{\delta} + \frac{1}{2}\,\lambda'\sigma\sigma'\lambda - \frac{1}{2}\,(\lambda' + B'_n)'\,\sigma\sigma'\,(\lambda' + B'_n) \tag{2.11}$$

$$B'_{n+1} = (\gamma' + B'_n\,\Phi). \tag{2.12}$$

APPENDIX II: DATA AND ESTIMATION

1. Data Sources and Handling

Yield data is made available by the Deutsche Bundesbank and can be obtained from their website. Data on *output*, *CPI inflation* and the *commodity price index* is taken from the *International Financial Statistics* provided by the International Monetary Fund.

The so-called *price norm* was announced annually by the Deutsche Bundesbank. Until 1984, the price norm, or price assumption, reflected the Bundesbank's view of the "unavoidable" level of inflation, while from 1985 onwards, it was defined as the maximum rate of inflation to be tolerated over the medium term. Conceptually, the price norm should refer to the GNP/GDP deflator rather the CPI, since it was related to the price term in the quantity equation. However, we believe it is a good approximation for the implicit target of the consumer price inflation. For the estimation the annual figures on the price norm have been interpolated into 12 monthly values. Whenever ranges instead of values were announced, the middle of the range was taken as a proxy. The following Table provides the values announced by the Deutsche Bundesbank.

Price Norm of the Deutsche Bundesbank 1979-1998

Year	*Price Norm*	*Year*	*Price Norm*
1979	4.0	1989	2.0
1980	3.5 – 4.0	1990	2.0
1981	3.5	1991	2.0
1982	3.5	1992	2.0
1983	3.0	1993	2.0
1984	2.0	1994	2.0
1985	2.0	1995	2.0
1986	2.0	1996	2.0
1987	2.0	1997	1.5 – 2.0
1988	2.0	1998	1.5 – 2.0

Source: Deutsche Bundesbank, monthly reports

The complete data file can be obtained from the author upon request.

2. Software and Estimation Details

The single equation estimations of the monetary policy rules in Table 2 were performed with the software package *Eviews*. The starting values for the GMM estimation were obtained from a previous OLS regression. However, the results remained stable when the starting values were varied. In the GMM estimation.we correct for heteroscedasticity and autocorrelation of unknown form with a lag truncation parameter of 12. In addition, we chose Bartlett weights to ensure positive definiteness of our estimated variance-covariance matrix. The lag structure of the instruments is as follows. We use the first six, the ninth and the twelfth lag of the output gap, inflation gap and the IMF commodity price index as well as the first, sixth, ninth and twelfth lag of the short term interest rate. This is close to the instruments suggested by Clarida, Galí and Gertler (1998).

Codes for the Kalman filter were programmed in Matlab. The codes can be provided by the author upon request. For all estimations the starting values for the parameters were computed as those minimizing the squared sum of residuals between estimated yields (with the factors being zero) and the mean observed yields. This minimization was performed using the add-in solver in Excel. For the calculation of the standard errors the 'finite difference method' was applied, thus, their were numerically computed on estimates of the inverse of the Hessian matrix at the convergence point. The Matlab codes can be provided upon request as well.